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# On lattices with Elkan's formula

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## Abstract

In this note we consider properties of lattices satisfying a special equation called Elkan's formula (E) :  $(x \wedge y')' = y \vee (x' \wedge y')$ . We show that, for any lattice  $L$  with a unary operator  $'$  and a greatest element 1, if it satisfies (E) then  $L^* = \{x'' \mid x \in L\}$  is a Boolean algebra.

## 1 Introduction

It is proved in [3] that each of de Morgan algebras and of orthomodular lattices with the formula  $(x \wedge y')' = y \vee (x' \wedge y')$  is a Boolean algebra. Since the equation is presented in the theory of fuzzy logic by C.Elkan ([2]), we call it here the *Elkan's formula* and denote it by (E). The results obtained in [3] mean that Boolean algebras are characterized by de Morgan algebras and orthomodular lattices by use of this equation. It is also proved in [4] that every orthocomplemented lattice with (E) is a Boolean algebra. In this note we give a more general result than [4], that is, for any lattice with a greatest element 1, if it satisfies only the condition

$$(E) \quad (x \wedge y')' = y \vee (x' \wedge y'),$$

then  $L^* = \{x'' \mid x \in L\}$  is a Boolean algebra. This implies a new characterization theorem of Boolean algebras.

## 2 Preliminaries

Let  $\mathcal{L} = (L, \wedge, \vee, ', 0, 1)$  be a bounded lattice with a unary operator  $'$ . A unary operator  $' : L \rightarrow L$  is called an *orthocomplementation* when it satisfies, for all  $x, y \in L$ ,

1.  $x \wedge x' = 0$ ;
2.  $x \vee x' = 1$ ;
3.  $x \leq y \implies y' \leq x'$ ;
4.  $x'' = x$ .

By an orthocomplemented lattice, we mean the bounded lattices with an orthocomplementation. If an orthocomplemented lattice  $\mathcal{L}$  satisfies the following conditions

$$(D1) (x \wedge y)' = x' \vee y', (x \vee y)' = x' \wedge y'$$

$$(D2) x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

then it is called a *de Morgan algebra*. An orthocomplemented lattice  $\mathcal{L}$  is called an *orthomodular lattice* when it satisfies

$$(D1) (x \wedge y)' = x' \vee y', (x \vee y)' = x' \wedge y'$$

$$(OM) x \leq y \implies y = x \vee (x' \wedge y) \quad (\text{Orthomodular law})$$

It is proved in [3, 4] that

1. The only de Morgan algebras in which the law (E)  $(x \wedge y)' = y \vee (x' \wedge y')$  holds are those that are Boolean algebras.
2. Every orthomodular lattice satisfying (E) is a Boolean algebra.
3. Every orthocomplemented lattice with (E) is a Boolean algebra.

### 3 Stronger result

We give a stronger result than [3, 4] in this section. Let  $\mathcal{L} = (L; \wedge, \vee, 0, 1)$  be a bounded lattice with a unary operator  $'$  satisfying (E)  $(x \wedge y)' = y \vee (x' \wedge y')$ . It is easy to prove the following results.

**Proposition 1.** *For all  $x, y \in L$ , we have*

1.  $0' = 1$
2.  $1'' = 1$
3.  $x \vee x' = 1$
4.  $x \leq x''$
5.  $x \leq y'$  if and only if  $y \leq x'$
6. if  $x \leq y$  then  $y' \leq x'$
7.  $x' = x'''$
8.  $(x \vee y)' = x' \wedge y'$

**Theorem 1.** *Let  $\mathcal{L} = (L; \wedge, \vee, 0, 1)$  be a bounded lattice. If a unary operator  $'$  satisfies only the following two conditions, then  $L$  is a Boolean algebra : For all  $x, y \in L$ ,*

$$(c1) \quad 1' = 0$$

$$(E) \quad (x \wedge y)' = y \vee (x' \wedge y')$$

Thus, the class of all bounded lattices with (c1) and (E) coincides with the class of Boolean algebras.

*Proof.* In order to prove our statement, it is enough to verify that the unary operator ' is an orthocomplementation. That is, we have to prove  $x \wedge x' = 0$  and  $x'' = x$ . At first we show that  $x'' = x$ . In the equation (E), if we take  $x = 1$  and  $y = x$  simultaneously, then we have

$$(1 \wedge x')' = x \vee (1' \wedge x').$$

Since  $1 \wedge x' = x'$  and  $1' \wedge x' = 0 \wedge x' = 0$  by (c1), it follows that

$$x'' = (1 \wedge x')' = x \vee (1' \wedge x') = x \vee 0 = x.$$

We note from the above that

$$0' = 1'' = 1.$$

Since  $1 = 0' = (0 \wedge x')' = x \vee (0' \wedge x') = x \vee x'$ , if we take  $y = x$  in (E), then  $(x \wedge x')' = x \vee (x' \wedge x') = x \vee x' = 1$ . It follows that

$$x \wedge x' = (x \wedge x')'' = 1' = 0.$$

□

As to the condition (c1) we can show the next result.

**Proposition 2.** *Let  $L$  be a bounded lattice with (E). Then the following statements are equivalent:*

1.  $1' = 0$
2.  $x = x''$  for all  $x \in L$
3.  $x \wedge x' = 0$  for all  $x \in L$
4. if  $x' \leq y$ , then  $y' \leq x$  for all  $x, y \in L$

*Proof.* 1.  $\implies$  2. : Suppose that  $1' = 0$ . For every element  $x \in L$ , we have

$$x'' = (1 \wedge x')' = x \vee (1' \wedge x') = x \vee (0 \wedge x') = x \vee 0 = x.$$

2.  $\implies$  3. : We suppose that  $x = x''$  for every  $x \in L$ . We note that  $0' = 1$  by (E). Indeed, if we take  $x = 0$  and  $y = x$  in the equation (E) then for all  $y \in L$  we have

$$0' = (0 \wedge y')' = y \vee (0' \wedge y') \geq y.$$

This means that  $0' = 1$  and  $1' = 0'' = 0$ . It follows from the above that  $1 = 0' = (0 \wedge x')' = x \vee (0' \wedge x') = x \vee (1 \wedge x') = x \vee x'$ . Thus we have

$$x \wedge x' = (x \wedge x')'' = (x \vee (x' \wedge x'))' = (x \vee x')' = 1' = 0.$$

3.  $\implies$  4. : We assume that  $x \wedge x' = 0$  for every  $x \in L$  and  $x' \leq y$ . It follows from the assumption that  $1' = 0$ . Then for all  $x \in L$ , since  $x'' = (1 \wedge x')' = x \vee (1' \wedge x') = x \vee (0 \wedge x') = x \vee 0 = x$  and  $x' \wedge y' = 0$  by  $x' \leq y$ , we have

$$x \wedge y' = (x \wedge y')'' = (y \vee (x' \wedge y'))' = (y \vee 0)' = y'.$$

This means that  $y' \leq x$ .

4.  $\implies$  1. : Assume that if  $x' \leq y$  then  $y' \leq x$  for all  $x, y \in L$ . Since  $x' \leq 1$  for every  $x \in L$ , we have  $1' \leq x$  by the assumption. Thus we have  $1' = 0$ .  $\square$

## 4 Substructure $L^*$

For any lattice  $\mathcal{L} = (L; \wedge, \vee)$  with a unary operator  $'$  and the Elkan's formula (E), we put  $L^* = \{x'' \mid x \in L\}$ . Since  $L^*$  is a subset of  $L$ , it has a partial order  $\leq^*$  which is a restriction of  $\leq$  on  $L^*$ . Thus, for all  $x, y \in L^*$  we have

$$x \leq^* y \iff x \leq y \text{ in } L$$

As to the order  $\leq^*$ , we have a following result.

**Proposition 3.**  $\mathcal{L}^* = (L^*, \leq^*)$  is also a lattice with a unary operator  $'$ , that is, for all  $x, y \in L^*$

$$\begin{aligned} \inf_{L^*} \{x, y\} &= x \wedge y \\ \sup_{L^*} \{x, y\} &= (x \vee y)'' \end{aligned}$$

*Proof.* We only show the case of "sup". It is obvious that  $x, y \leq x \vee y \leq (x \vee y)''$ . For every  $u \in L^*$  such that  $x, y \leq^* u$ , since  $x, y \leq u$  and  $x \vee y \leq u$ , we have  $(x \vee y)'' \leq^* u'' = u$ . This means that  $\sup_{L^*} \{x, y\} = (x \vee y)''$ .  $\square$

We use symbols  $\sqcap$  and  $\sqcup$  for "inf" and "sup" in  $L^*$  respectively to avoid confusions: For  $x, y \in L^*$ ,

$$\begin{aligned} x \sqcap y &= \inf_{L^*} \{x, y\} \\ x \sqcup y &= \sup_{L^*} \{x, y\} \end{aligned}$$

We can show that the lattice  $L^*$  satisfies the *orthomodular law* : if  $a \leq^* b$  then  $b = a \sqcup (a' \sqcap b)$ .

**Proposition 4.** The lattice  $L^*$  satisfies the orthomodular law : if  $a \leq^* b$  then  $b = a \sqcup (a' \sqcap b)$ .

*Proof.* Suppose that  $a \leq^* b$  for  $a, b \in L^*$ . Since  $b = b''$  and  $b' \leq^* a'$ , we have

$$\begin{aligned} b &= b'' = (b' \wedge a')' \\ &= \{(b' \wedge a')'\}'' = \{a \vee (b'' \wedge a')\}'' \\ &= \{a \vee (b \wedge a')\}'' = a \sqcup (b \wedge a') \\ &= a \sqcup (a' \sqcap b). \end{aligned}$$

Thus, the lattice  $L^*$  satisfies the orthomodular law.  $\square$

Next we consider a case of  $L$  having a greatest element 1. We see that  $1'$  is the least element in  $L^*$ . Moreover we have the following result.

**Proposition 5.** *Let  $L$  be a lattice with a unary operator  $'$  satisfying the Elkan's formula (E). If there is a greatest element 1 in  $L$ , then we have for all  $x \in L$*

- (1)  $1 = 1''$ ;
- (2)  $x \vee x' = 1$ ;
- (3)  $x' \wedge x'' = 1'$ .

*Proof.* Suppose that  $L$  is a lattice with a unary operator  $'$  satisfying (E) and a greatest element 1.

(1) If we take  $x = y = 1$  in (E), then we have

$$1'' = (1 \wedge 1')' = 1 \vee (1' \wedge 1') \geq 1.$$

(2) Take  $x = 1', y = x$ . We get that

$$1 = 1'' = (1' \wedge x')' = x \vee (1'' \wedge x') = x \vee (1 \wedge x') = x \vee x'.$$

(3) It is trivial that  $x' \wedge x'' = (x \vee x')' = 1'$ . □

For any lattice  $L$  with meeting the orthomodular law, an element  $x \in L$  is said to be *commutative* with  $y$  (denoted by  $xCy$ ) if  $x = (x \wedge y) \vee (x \wedge y')$ . As to the commutativity, it is obvious that (1) if  $x \leq y$  then  $xCy$  and (2) if  $xCy$  then  $xCy'$ .

A set  $\{x, y, z\}$  of elements of  $L$  is called *distributive triple* if  $xCy$  and  $xCz$ . It is well-known ([1]) that if  $\{x, y, z\}$  is distributive triple then the sublattice generated by  $\{x, y, z\}$  is a distributive lattice (Foulis-Holland theorem), thus the following holds :

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ y \wedge (z \vee x) &= (y \wedge z) \vee (y \wedge x) \\ z \wedge (x \vee y) &= (z \wedge x) \vee (z \wedge y) \\ (x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z) \\ (y \wedge z) \vee x &= (y \vee x) \wedge (z \vee x) \\ (z \wedge x) \vee y &= (z \vee y) \wedge (x \vee y). \end{aligned}$$

From the fact above, we can conclude that

**Lemma 1.** *For any  $a, b \in L^*$ , we have  $aCb$ , that is,*

$$a = (a \sqcap b) \sqcup (a \sqcap b').$$

*Hence  $L^*$  is a distributive lattice.*

*Proof.* Since the orthomodular law holds in the lattice  $L^*$  and  $a \sqcap b' \leq^* a$ , we have

$$\begin{aligned} a &= (a \sqcap b') \sqcup \{(a \sqcap b')' \sqcup a\} \\ &= (a \sqcap b') \sqcup \{b \sqcup (a' \sqcap b')\} \sqcup a. \end{aligned}$$

The fact that  $a' \sqcap b' \leq^* a', b'$  implies that  $(a' \sqcap b')Ca'$  and  $(a' \sqcap b')Cb'$ . Hence we get  $(a' \sqcap b')Ca$  and  $(a' \sqcap b')Cb$  by  $a = a''$  and  $b'' = b$  for  $a, b \in L^*$ . This means that a set  $\{a, b, a' \sqcap b'\}$  is distributive triple. From the Foulis-Holland theorem, it follows that

$$\begin{aligned} a \sqcap \{b \sqcup (a' \sqcap b')\} &= (a \sqcap b) \sqcup (a \sqcap a' \sqcap b') \\ &= (a \sqcap b) \sqcup (1' \sqcap b) \\ &= (a \sqcap b) \sqcup 1' \\ &= a \sqcap b, \end{aligned}$$

because  $1'$  is the least element in  $L^*$ . This implies that

$$a = (a \sqcap b') \sqcup (a \sqcap b).$$

That is, for all elements  $a, b \in L^*$ , we have  $aCb$ . Thus  $L^*$  is a distributive lattice.  $\square$

Moreover, we see that each element  $a \in L^*$  has a complement  $a' \in L^*$  from the proposition above.

**Theorem 2.** *If a lattice  $L$  has a greatest element 1 and satisfies the Elkan's formula, then the lattice  $\mathcal{L}^* = (L^*, \sqcap, \sqcup, ', 1', 1)$  is a maximal Boolean algebra included in  $L$ .*

*Proof.* We only show its maximality. If  $L^* \subset B$  for some Boolean algebra included in  $L$ , then there is an element  $x \in B$  such that  $x \notin L^*$ . Since  $B$  is the Boolean algebra, we have  $x = x''$  and thus  $x \in L^*$ . This is a contradiction. Hence  $L^*$  is the maximal Boolean algebra included in  $L$ .  $\square$

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